Introduction to optimization

Linear Programming

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Outline

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- Content
- What is optimization

Linear programming problem
- An example
- The simplex method

Content of the course

- Introduction the problem, motivation
- Linear programming
- Mathematical modeling
- The Simplex method
- Dual formulation
- Links with graph theory
- Integer programming
- Transport problems
- Introduction to non-linear programming

General form

Cost function and constraints

An optimization problem generally has the following form

$$\begin{aligned} \text{minimize } & f_0(\mathsf{x}) \\ \text{subject to } & f_i(\mathsf{x}) \leq b_i, i = 1, \dots, m \end{aligned} \quad (1)$$

$\mathsf{x} = (x_1, \dots, x_n)$ is a vector of $\mathbb{R}^n$ called the *optimization variable* of the problem; $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the *cost function* functional; the $f_i : \mathbb{R}^n \to \mathbb{R}$ are the *constraints* and the b_i are the *bounds* (or limits).

A vector $\mathsf{x}^\star$ is is *optimal*, or is a solution to the problem, if it has the smallest objective value among all vectors that satisfy the constraints.

Some examples

- Cost minimisation tasks
- Task assignment
- Shipping problems
- Classical problems: The travelling salesman problem (TSP)
- Other NP-hard problems: knapsack, subset-sum, etc.
- Transport, flow networks, etc

The TSP

- Let a traveling salesperson having to go through N cities, all connected
- What trip will minimize the total distance travelled?
- NP problems are decision problem, the associated DP is: "Let there be N cities, all connected, does a trip of less than K km exist ?".
- We do not know of any fundamentally better method than the one which consists in trying all the possible trips
    - 2 cities : 1 trip (back and forth are equivalent)
    - 5 cities : 60 different trips
    - n cities : $n!/2$ trips.

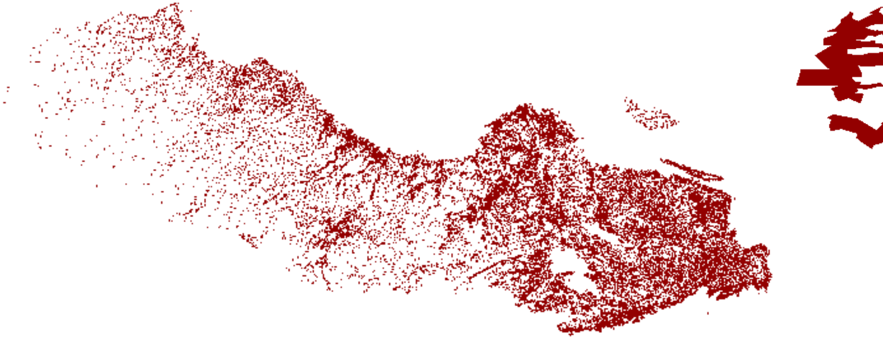
Example of practical TSP - PCB routing, 3038 “cities”

History of the TSP

Year	Team	Size of the TSP solved
1954	Dantzig, Fulkerson, Johnson	49
1971	Held, Karp	64
1977	Grötschel	120
1987	Padberg, Rinaldi	2392
2004	Applegate et al.	24978
2006	Applegate et al..	85900

Example: tour of Sweden

(North is left)



Resources allocation

- Given a problem, allocate resources in the best possible way.
- Very common: money, materials, personnel, time, etc
- Some problems have an optimal solution, others do not.
- A non-trivial class of important problems with a computable solution are the *linear programming* problems

Dog food

- Let there be a company manufacturing dog food. They make two products: Wag-Tail (W) and Bark-Mad (B).
- Each of those item uses a mix of Vegetables, Beef and Fish, in the following proportions:

Ingredient	Total Qty	Qty in B	Qty in W
Vegetables	1400 kg	4 kg	4 kg
Fish	1800 kg	6 kg	3 kg
Beef	1800 kg	2 kg	6 kg

- We assume that the company makes a benefit of 12 euros on each packet of B and 8 euros on each packet of W.
- What should the company do to maximize its profit?

Formulation

- Let B be the number of Bark-Mad products, and W the number of Wag-Tail products.
- From the table previously, we deduce that the total quantity of vegetables will be $4W + 4B$ kg, but we cannot consume more than 1400 kg of vegetables, therefore :

$$4B + 4W \leq 1400 \quad (2)$$

Formulation II - constraints

- In the same way :

$$6B + 3W \leq 1800 \quad (3)$$

- and

$$2B + 6W \leq 1800 \quad (4)$$

- Also, very importantly B et W are both positive.

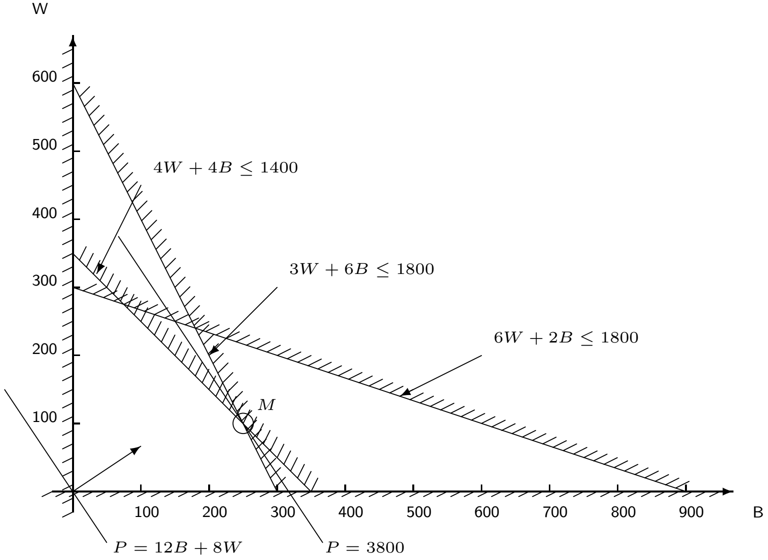
Formulation III - objective

- The total profit is:

$$P = 12B + 8W \tag{5}$$

- We want to maximize P
- We have only two variables, so we can make a drawing

Dog food problem, graphical form



Solution

- We move a line from the origin in a parallel fashion to itself, with an equation equal to P.
- We stop when we maximize P while still satisfying constraints
- Here the solution is $B = (250, 100)$ – At this point two resources are fully consumed.

Vocabulary & notes

- A solution is *feasible* if it is a solution, and satisfies all the constraints ;
- The set of all feasible solutions is a *convex polytope* (2D: polygon, 3D: polyhedron, nD: polytope)
- The optimal solution includes at least one vertex of the polytope ;
- To find the optimal solution, it is therefore enough to explore all the vertices of the polytope.
- Note that this set is finite.

Possible outcome of an LP problem

1. The solution exists and is unique (as before) ;
2. a solution exists but is not unique ;
3. there is no solution ;
4. the solution is unbounded.

History

- In 1947, G. B. Dantzig was working as a civilian at the Pentagone as a mathematical expert;
- He has planning problems to solve (Planes, personnel, etc);
- After talking with economist T. Koopmans, he realized there did not exist any methodology (in the West) to find a solution to these problems;
- Dantzig so looked for one.

Basic idea

- In more than 3 dimensions, a geometrical object equivalent to a polygon in 2D and a polyhedra in 3D is called a *Polytope* in geometry, and a *simplex* in algebraic topology.
- If the solution is on the vertices of a polytope, one can start from a feasible solution (on the polytope), and then move along the edges until we find an optimal vertex.
- Dantzig did not initially think it would be efficient, but in fact one can find the optimal solution in about as many moves as dimensions in the space of solutions, which is very efficient.

The simplex method

1. From a feasible solution, calculate the solution at this point;
2. find the edges stemming from this vertex. Compute if the objective function improves along one or more of the edges;
3. move along the edge yielding the highest improvement;
4. Repeat (2) et (3) until the objective function does not improve anymore. This is the optimal solution.

Dog food example

- We eliminate inequalities with 3 artificial variables (slack variables) s_i.
- We maximize $P = 12B + 8W$
- Under the constraints :

$$4B + 4W + s_1 = 1400 \tag{6}$$
$$6B + 3W + s_2 = 1800 \tag{7}$$
$$2B + 6W + s_3 = 1800 \tag{8}$$

Terminology

- The set of all values of B, W et s_i make up a *feasible solution*
- A solution with m equations and m unknowns with some of these variables at zero is a *basis solution*
- The set of m non-zero variables forms a *basis*.

Initial Choice

- To form a basis solution, one must choose 3 variables, the other being set to 0.
- We can choose $B = W = 0$ and solve for all the s_i
- This yields:

$$s_1 = 1400 - 4B - 4W \tag{9}$$
$$s_2 = 1800 - 6B - 3W \tag{10}$$
$$s_3 = 1800 - 2B - 6W \tag{11}$$
$$P = 12B + 8W \tag{12}$$

- Geometrically, this solution is feasible and corresponds to the origin. Unfortunately, for this choice, $P = 0$, which is not optimal.

First iteration

- To augment P, we can choose to increase B or W. Since B provides the biggest gain, we choose this variable. W stays at zero.
- B is introduced in the basis. However, B cannot be larger than 300, otherwise s_2 would become negative. So we choose $B = 300$, which induces $s_2 = 0$.
- We re-express the system with the in-base variable as a function of the out-of bases variables:
  The second equation becomes:

$$
\begin{aligned}
s_2 &= 1800 - 6B - 3W \\
-6B &= 1800 - s_2 - 3W \\
B &= 300 - \frac{1}{6}s_2 - \frac{1}{2}W
\end{aligned}
$$

Result of the first iteration

- Subsituting the B identity in the first equation:

$$s_1 = 1400 - 4(300 - \frac{1}{6}s_2 - \frac{1}{2}W) - 4W = 200 + \frac{2}{3}s_2 - 2W$$

- In the same way for the third equation and the objective function, we obtain:

$$B = 300 - \frac{1}{6}s_2 - \frac{1}{2}W \quad (13)$$

$$s_1 = 200 + \frac{2}{3}s_2 - 2W \quad (14)$$

$$s_3 = 1200 + \frac{1}{3}s_2 - 5W \quad (15)$$

$$P = 3600 - 2s_2 + 2W \quad (16)$$

- Now, with s_2 and W both at zero, profit is 3600.

Second iteration

- We can still improve things by augmenting W, while maintaining s_1 positive. We find that W cannot be larger than 100.
- So we introduce W in the basis. With $W = 100$, this results in $s_1 = 0$.
- We re-express the system again as a function of s_1 and s_2

$$W = 100 - \frac{1}{2}s_1 + \frac{1}{3}s_2 \quad (17)$$

$$B = 250 + \frac{1}{4}s_1 - \frac{1}{3}s_2 \quad (18)$$

$$s_3 = 700 + \frac{5}{2}s_1 - \frac{4}{3}s_2 \quad (19)$$

$$P = 3800 - s_1 - \frac{4}{3}s_2 \quad (20)$$

- Profit is now 3800 with s_1 and s_2 at zero.

Optimum

- We cannot improve P further, since augmenting s_1 or s_2 would decrease P.
- We can interpret the fact that s_1 (vegetables) et s_2 (fish) are set to 0 as the fact that these resources are completely consumed. This is not the case for s_3 (beef). There is some leftover beef in the end, which cannot be used.

Summary

- The simplex method is an algorithm for solving Linear Programming problems.
- The method is simple and has an intuitive geometrical interpetation.
- In practice, this algorithm finds the optimum in a few iterations when it exists.
- In the remainder, we shall explore the algorithm in detail, including limit cases.